

Strong monogamy of quantum entanglement for multi-qubit W-class states

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We provide a strong evidence for strong monogamy inequality of multi-qubit entanglement recently proposed in [B. Regula *et al.*, Phys. Rev. Lett. **113**, 110501 (2014)]. We consider a large class of multi-qubit generalized W-class states, and analytically show that the strong monogamy inequality of multi-qubit entanglement is saturated by this class of states.

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I. INTRODUCTION

Whereas classical correlation can be freely shared among parties in multi-party systems, quantum entanglement is restricted in its shareability; if a pair of parties are maximally entangled in multipartite systems, they cannot have any entanglement [1, 2] nor classical correlations [3] with the rest of the system. This restriction of entanglement shareability among multi-party systems is known as the *monogamy of entanglement* (MoE) [4].

MoE is at the heart of many quantum information and communication protocols. For example, MoE is a key ingredient to make quantum cryptography secure because it quantifies how much information an eavesdropper could potentially obtain about the secret key to be extracted [5]. MoE also plays an important role in condensed-matter physics such as the frustration effects observed in Heisenberg antiferromagnets and the N -representability problem for fermions [6].

The first mathematical characterization of MoE was established by Coffman-Kundu-Wootters (CKW) for three-qubit systems [1] as an inequality; for a three-qubit pure state $|\psi\rangle_{ABC}$ with its one-qubit and two-qubit reduced density matrices $\rho_A = \text{tr}_{BC}|\psi\rangle_{ABC}\langle\psi|$, $\text{tr}_C|\psi\rangle_{ABC}\langle\psi| = \rho_{AB}$ and $\text{tr}_B|\psi\rangle_{ABC}\langle\psi| = \rho_{AC}$ respectively,

$$\tau(|\psi\rangle_{A|BC}) \geq \tau(\rho_{A|B}) + \tau(\rho_{A|C}), \quad (1)$$

where $\tau(|\psi\rangle_{A|BC})$ is the *tangle* of the pure state $|\psi\rangle_{ABC}$ quantifying the bipartite entanglement between A and BC , and $\tau(\rho_{A|B})$ and $\tau(\rho_{A|C})$ are the tangles of the two-qubit reduced states $\rho_{AB} = \text{tr}_C|\psi\rangle_{ABC}\langle\psi|$ and $\rho_{AC} = \text{tr}_B|\psi\rangle_{ABC}\langle\psi|$, respectively.

CKW inequality in (1) shows the mutually exclusive nature of three-qubit quantum entanglement in a quantitative way; more entanglement shared between two qubits ($\tau(\rho_{A|B})$) necessarily implies less entanglement between the other two qubits ($\tau(\rho_{A|C})$) so that the summation does not exceed the total entanglement ($\tau(|\psi\rangle_{A|BC})$). Moreover, the residual entanglement

from the difference between left and right-hand sides of Inequality (1) is also interpreted as the genuine three-party entanglement, *three tangle* of $|\psi\rangle_{ABC}$

$$\tau(|\psi\rangle_{A|B|C}) = \tau(|\psi\rangle_{A|BC}) - \tau(\rho_{A|B}) - \tau(\rho_{A|C}), \quad (2)$$

which is invariant under the permutation of subsystems A , B and C . In this sense, $\tau(|\psi\rangle_{A|BC})$ and $\tau(\rho_{A|B})$ are also referred to as the one tangle and two tangle, respectively [7].

Later, CKW inequality was generalized for multi-qubit systems [2] and some cases of higher-dimensional quantum systems in terms of various entanglement measures [8–11]. For general monogamy inequality of multi-party entanglement, it was shown that squashed entanglement [12] is a faithful entanglement measure showing MoE of arbitrary quantum systems [13].

Recently, the three-tangle in Eq. (2) was systematically generalized for arbitrary n -qubit quantum states, namely residual n -tangle [14]. Based on this generalization, the concept of *strong monogamy* (SM) inequality of multi-qubit entanglement was proposed by conjecturing the nonnegativity of the n -tangle [14]. For the validity of SM inequality, an extensive numerical evidence was presented for four qubit systems together with analytical proof for some cases of multi-qubit systems. However, proving SM conjecture analytically for arbitrary multi-qubit states seems to be a formidable challenge due to the numerous optimization processes arising in the definition of n -tangle.

Here we provide a strong evidence for SM inequality of multi-qubit entanglement; we consider a large class of multi-qubit states, *generalized W-class states*, and analytically show that SM inequality proposed in [14] is saturated by this class of states. Because multi-qubit CKW inequality is known to be saturated by this generalized W-class states [15], this class of states are good candidates as possible counterexamples for stronger version of monogamy inequalities.

The paper is organized as follows. In Sec. II, we review the definition of generalized W-class states for multi-qubit systems and provide some useful properties of this class in accordance with CKW inequality. In Sec. III A, we recall the concept of n -tangle as well as SM inequality of multi-qubit entanglement, and show that multi-qubit

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SM inequality is saturated by generalized W-class states in Sec. III B. In Sec. IV, we summarize our results.

II. MULTI-QUBIT CKW INEQUALITY AND THE GENERALIZED W-CLASS STATES

For n -qubit systems $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ where $\mathcal{H}_j \cong \mathbb{C}^2$ for $j = 1, \dots, n$ and any n -qubit state $|\psi\rangle_{A_1 A_2 \dots A_n} \in \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$, the three-qubit CKW inequality in (1) can be generalized as [2]

$$\tau(|\psi\rangle_{A_1 A_2 \dots A_n}) \geq \sum_{j=2}^n \tau(\rho_{A_1 | A_j}), \quad (3)$$

where $\tau(|\psi\rangle_{A_1 A_2 \dots A_n})$ is the tangle (or one tangle) of the pure state $|\psi\rangle_{A_1 A_2 \dots A_n}$ with respect to the bipartition between A_1 and the other qubits $A_2 \cdots A_n$

$$\tau(|\psi\rangle_{A_1 A_2 \dots A_n}) = 4 \det \rho_A, \quad (4)$$

and $\tau(\rho_{A_1 | A_j})$ is the tangle (or two tangle) of the two-qubit reduced density matrix $\rho_{A_1 A_j}$ defined by convex-roof extension

$$\tau(\rho_{A_1 | A_j}) = \left[\min_h \sum_h p_h \sqrt{\tau(|\psi_h\rangle_{A_1 A_j})} \right]^2, \quad (5)$$

with the minimization taken over all possible pure state decompositions

$$\rho_{A_1 A_j} = \sum_h p_h |\psi_h\rangle_{A_1 A_j} \langle \psi_h|, \quad (6)$$

for each $j = 2, \dots, n$.

The n -qubit the generalized W-class state is defined as

$$|\psi\rangle_{A_1 A_2 \dots A_n} = a|00 \cdots 0\rangle + b_1|10 \cdots 0\rangle + b_2|01 \cdots 0\rangle + \dots + b_n|00 \cdots 1\rangle \quad (7)$$

with $|a|^2 + \sum_{i=1}^n |b_i|^2 = 1$ [15, 16]. The term “*generalized*” naturally arises because Eq. (7) includes n -qubit W states as a special case when $a = 0$ and $b_j = 1/\sqrt{n}$ for all j .

Before we further investigate strongly monogamous property of entanglement for the generalized W-class state in Eq. (7), we recall a very useful property of quantum states proposed by Hughston-Jozsa-Wootters (HJW), which shows the unitary freedom in the ensemble for density matrices [17].

Proposition 1. (*HJW theorem*) The sets $\{|\tilde{\phi}_i\rangle\}$ and $\{|\tilde{\psi}_j\rangle\}$ of (possibly unnormalized) states generate the same density matrix if and only if

$$|\tilde{\phi}_i\rangle = \sum_j u_{ij} |\tilde{\psi}_j\rangle \quad (8)$$

where (u_{ij}) is a unitary matrix of complex numbers, with indices i and j , and we pad whichever set of states $\{|\tilde{\phi}_i\rangle\}$ or $\{|\tilde{\psi}_j\rangle\}$ is smaller with additional zero vectors so that the two sets have the same number of elements.

A direct consequence of Proposition 1 is the following; for two pure-state decompositions $\sum_i p_i |\phi_i\rangle \langle \phi_i|$ and $\sum_j q_j |\psi_j\rangle \langle \psi_j|$, they represent the same density matrix, that is $\rho = \sum_i p_i |\phi_i\rangle \langle \phi_i| = \sum_j q_j |\psi_j\rangle \langle \psi_j|$ if and only if $\sqrt{p_i} |\phi_i\rangle = \sum_j u_{ij} \sqrt{q_j} |\psi_j\rangle$ for some unitary matrix u_{ij} . Now we have the following lemma showing that multi-qubit monogamy inequality in terms of one and two tangles in (3) is saturated by the generalized W-class states in (7).

Lemma 1. For any multi-qubit system, multi-qubit CKW inequality is saturated by generalized W-class states, that is,

$$\tau(|\psi\rangle_{A_1 A_2 \dots A_n}) = \sum_{j=2}^n \tau(\rho_{A_1 | A_j}), \quad (9)$$

for any n -qubit state $|\psi\rangle_{A_1 A_2 \dots A_n}$ in Eq. (7)

Proof. Let us first consider the one tangle of $|\psi\rangle_{A_1 A_2 \dots A_n}$ with respect to the bipartition between A_1 and the other qubits. The reduced density matrix ρ_{A_1} of subsystem A_1 is

$$\begin{aligned} \rho_{A_1} &= \text{tr}_{A_2 \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= (a|0\rangle + b_1|1\rangle)_{A_1} (a^* \langle 0| + b_1^* \langle 1|) + \sum_{j=2}^n |b_j|^2 |0\rangle_{A_1} \langle 0|, \end{aligned} \quad (10)$$

thus

$$\tau(|\psi\rangle_{A_1 A_2 \dots A_n}) = 4 \det \rho_{A_1} = 4|b_1|^2 \sum_{j=2}^n |b_j|^2. \quad (11)$$

For each $j = 2, 3, \dots, n$, the reduced density matrix $\rho_{A_1 A_j}$ of two-qubit subsystem $A_1 A_j$ is

$$\begin{aligned}\rho_{A_1 A_j} &= \text{tr}_{A_2 \dots \widehat{A_j} \dots A_n} |\psi\rangle_{A_1 A_2 \dots A_n} \langle \psi| \\ &= (a|00\rangle + b_1|10\rangle + b_j|01\rangle)_{A_1 A_j} (a^*\langle 00| + b_1^*\langle 10| + b_j^*\langle 01|) + \sum_{k \neq j} |b_k|^2 |00\rangle_{A_1 A_j} \langle 00|,\end{aligned}\quad (12)$$

where $A_2 \dots \widehat{A_j} \dots A_n = A_2 \dots A_{j-1} A_{j+1} \dots A_n$ for each $j = 2, 3, \dots, n$. Here, we consider two-qubit (possibly) unnormalized states

$$\begin{aligned}|\tilde{x}\rangle_{A_1 A_j} &= a|00\rangle_{A_1 A_j} + b_1|10\rangle_{A_1 A_j} + b_j|01\rangle_{A_1 A_j} \\ |\tilde{y}\rangle_{A_1 A_j} &= \sqrt{\sum_{k \neq j} |b_k|^2} |00\rangle_{A_1 A_j},\end{aligned}\quad (13)$$

which represents $\rho_{A_1 A_j}$ as

$$\rho_{A_1 A_j} = |\tilde{x}\rangle_{A_1 A_j} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_j} \langle \tilde{y}|. \quad (14)$$

From the HJW theorem in Proposition 1, we note that for any pure state decomposition of

$$\rho_{A_1 A_j} = \sum_{h=1}^r |\tilde{\phi}_h\rangle_{A_1 A_j} \langle \tilde{\phi}_h|, \quad (15)$$

where $|\tilde{\phi}_h\rangle_{A_1 A_j}$ is an unnormalized state in two-qubit subsystem $A_1 A_j$, there exists an $r \times r$ unitary matrix (u_{hl}) such that

$$|\tilde{\phi}_h\rangle_{A_1 A_j} = u_{h1} |\tilde{x}\rangle_{A_1 A_j} + u_{h2} |\tilde{y}\rangle_{A_1 A_j}, \quad (16)$$

for each h .

By considering the normalization $|\phi_h\rangle_{A_1 A_j} = |\tilde{\phi}_h\rangle_{A_1 A_j} / \sqrt{p_h}$ with $p_h = |\langle \tilde{\phi}_h | \tilde{\phi}_h \rangle|$, we have the tangle of each two-qubit pure state $|\phi_h\rangle_{A_1 A_j}$ as

$$\tau(|\phi_h\rangle_{A_1 A_j}) = 4 \det \rho_{A_1}^h = \frac{4}{p_h^2} |u_{hj}|^4 |b_1|^2 |b_j|^2, \quad (17)$$

where $\rho_{A_1}^h = \text{tr}_{A_j} |\phi_h\rangle_{A_1 A_j} \langle \phi_h|$ is the reduced density matrix of $|\phi_h\rangle_{A_1 A_j}$ on subsystem A_1 for each h . Moreover, the definition of two-tangle in Eq. (5) together with Eq. (17) lead us to

$$\begin{aligned}\tau(\rho_{A_1 A_j}) &= \left[\min_{\{p_h, |\phi_h\rangle\}} \sum_h p_h \sqrt{\tau(|\phi_h\rangle_{A_1 A_j})} \right]^2 \\ &= \left[\min_{\{p_h, |\phi_h\rangle\}} \sum_h 2|u_{hj}|^2 |b_1| |b_j| \right]^2 \\ &= 4|b_1|^2 |b_j|^2,\end{aligned}\quad (18)$$

for each $j = 2, \dots, n$.

Now Eqs. (11) and (18) implies Eq. (9), which completes the proof. \square

For two tangle of two-qubit mixed state $\rho_{A_1 A_j}$ in Eq. (12), we need to deal with the minimization arising in the definition Eq. (5). In fact, any two-qubit mixed state can have an analytic entanglement measure called *concurrence* [18], whose analytic evaluation can also be adapted for that of two tangle. However, the proof of Lemma 1 efficiently resolves this optimization problem by considering all possible pure-state decompositions of $\rho_{A_1 A_j}$, which also shows a nice property of generalized W-class states; the tangle of two-qubit reduced density matrix obtained from a generalized W-class state does not depend on the choice of pure-state decomposition, $\rho_{A_1 A_j} = \sum_h p_h |\phi_h\rangle_{A_1 A_j} \langle \phi_h|$. The following simple lemma shows another useful property about the structure of generalized W-class.

Lemma 2. *Let $|\psi\rangle_{A_1 \dots A_n}$ be a generalized W-class state in Eq. (7). For any m -qubit subsystems $A_1 A_{j_1} \dots A_{j_{m-1}}$ of $A_1 \dots A_n$ with $2 \leq m \leq n-1$, the reduced density matrix $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ of $|\psi\rangle_{A_1 \dots A_n}$ is a mixture of a m -qubit generalized W-class state and vacuum.*

Proof. By a straightforward calculation, we obtain

$$\begin{aligned}\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} &= |\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{x}| \\ &\quad + |\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{y}|,\end{aligned}\quad (19)$$

where

$$\begin{aligned}
|\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} &= (a|00\dots 0\rangle + b_1|10\dots 0\rangle + b_{j_1}|01\dots 0\rangle + \dots + b_{j_{m-1}}|00\dots 1\rangle)_{A_1 A_{j_1} \dots A_{j_{m-1}}}, \\
|\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} &= \sqrt{\sum_{k \in \{j_1, j_2, \dots, j_{m-1}\}} |b_k|^2} |00\dots 0\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}
\end{aligned} \tag{20}$$

are the unnormalized states in m -qubit subsystems $A_1 A_{j_1} \dots A_{j_{m-1}}$. By considering the normalized states $|x\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} = |\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} / \sqrt{p}$ with $p = \langle \tilde{x} | \tilde{x} \rangle$ and $|y\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} = |\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} / \sqrt{q}$ with $q = \langle \tilde{y} | \tilde{y} \rangle$, we note that

$$\begin{aligned}
\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} &= p|x\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle x| \\
&\quad + q|y\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle y|, \tag{21}
\end{aligned}$$

where $|x\rangle$ is a generalized W-class state and $|y\rangle$ is the vacuum, which completes the proof. \square

III. STRONG MONOGAMY INEQUALITY FOR MULTI-QUBIT GENERALIZED W-CLASS STATES

A. Strong monogamy of multi-qubit entanglement

The definition of three tangle in Eq. (2) was generalized for arbitrary n -qubit quantum states [14]; for an n -qubit

pure state $|\psi\rangle_{A_1 A_2 \dots A_n}$, its n -tangle is defined as

$$\begin{aligned}
\tau(|\psi\rangle_{A_1 | A_2 | \dots | A_n}) &= \tau(|\psi\rangle_{A_1 | A_2 \dots A_n}) \\
&\quad - \sum_{m=2}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1 | A_{j_1^m} | \dots | A_{j_{m-1}^m}}\right)^{m/2}, \tag{22}
\end{aligned}$$

where the index vector $\vec{j}^m = (j_1^m, \dots, j_{m-1}^m)$ spans all the ordered subsets of the index set $\{2, \dots, n\}$ with $(m-1)$ distinct elements. For each $m = 2, \dots, n-1$, the m -tangle for multi-qubit mixed state is defined by convex-roof extension,

$$\tau\left(\rho_{A_1 | A_{j_1^m} | \dots | A_{j_{m-1}^m}}\right) = \left[\min_{\{|\psi_h\rangle, |\psi_h\rangle\}} \sum_h p_h \sqrt{\tau\left(|\psi_h\rangle_{A_1 | A_{j_1^m} | \dots | A_{j_{m-1}^m}}\right)} \right]^2, \tag{23}$$

where the minimization is over all possible pure state decompositions

$$\rho_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}} = \sum_h p_h |\psi_h\rangle_{A_1 A_{j_1^m} \dots A_{j_{m-1}^m}} \langle \psi_h|. \tag{24}$$

Eq. (22) is a recurrent definition, that is, all the m tangles $\tau\left(\rho_{A_1 | A_{j_1^m} | \dots | A_{j_{m-1}^m}}\right)$ for $2 \leq m \leq n-1$ need to appear to define the n tangle $\tau(|\psi\rangle_{A_1 | A_2 | \dots | A_n})$. We further note that Eq. (22) reduces to the two and three tangles when $n = 2$ and $n = 3$ respectively. Based on this generalization, strong monogamy of multi-qubit entanglement was proposed by conjecturing the nonnegativity

of n -tangle Eq. (22),

$$\tau(|\psi\rangle_{A_1 | A_2 \dots A_n}) \geq \sum_{m=2}^{n-1} \sum_{\vec{j}^m} \tau\left(\rho_{A_1 | A_{j_1^m} | \dots | A_{j_{m-1}^m}}\right)^{m/2}. \tag{25}$$

The term *strong* naturally arises because Inequality (25) is in fact *finer* than the n -qubit CKW inequality

in (3)

$$\begin{aligned} \tau \left(|\psi\rangle_{A_1|A_2\cdots A_n} \right) &\geq \sum_{j=2}^n \tau \left(\rho_{A_1|A_j} \right) \\ &\quad + \sum_{m=3}^{n-1} \sum_{\vec{j}_m} \tau \left(\rho_{A_1|A_{j_1}^{j_1} \cdots |A_{j_{m-1}}^{j_{m-1}}} \right)^{m/2} \\ &\geq \sum_{j=2}^n \tau \left(\rho_{A_1|A_j} \right). \end{aligned} \quad (26)$$

Moreover, Inequality (25) also encapsulates three-qubit CKW inequality in (1) for $n = 3$, thus Inequality (25) can be considered as another generalization of three-qubit CKW inequality in a stronger form.

B. SM inequality for W-class states

For the validity of SM inequality in (25), an extensive numerical evidence was presented for four qubit systems together with analytical proof for some cases of multi-qubit systems. However, providing an analytical proof of Inequality (25) for arbitrary multi-qubit states seems to be a formidable challenge because there are numerous optimization processes arising in the recurrent definition of n -tangle (22). Here we show that SM inequality holds for generalized W-class states in arbitrary multi-qubit systems. Because Lemma 1 shows the multi-qubit CKW inequality is saturated by generalized W-class states [15], this class of states are good candidates for possible violation of stronger inequality, that is, SM inequality.

For the validity of SM inequality for generalized W-class states, we first note that Inequality (25) must be saturated by this class of states because of Lemma 1 together with Inequalities (26). Thus we will show the residual term

$$\sum_{m=3}^{n-1} \sum_{\vec{j}_m} \tau \left(\rho_{A_1|A_{j_1}^{j_1} \cdots |A_{j_{m-1}}^{j_{m-1}}} \right)^{m/2} \quad (27)$$

in (26) is zero for any n -qubit generalized W-class state $|\psi\rangle_{A_1 A_2 \cdots A_n}$. By using the mathematical induction on m , we further show that all the m tangles for $3 \leq m \leq n-1$ is zero for generalized W-class states, that is,

$$\tau \left(\rho_{A_1|A_{j_1}^{j_1} \cdots |A_{j_{m-1}}^{j_{m-1}}} \right) = 0, \quad (28)$$

for all the index vectors $\vec{j}^m = (j_1^m, \dots, j_{m-1}^m)$ with $3 \leq m \leq n-1$.

For $m = 3$ and any index vector $\vec{j} = (j_1, j_2)$ with $j_1, j_2 \in \{2, 3, \dots, n\}$, the left-hand side of Eq. (28) becomes the three-tangle of the three-qubit subsystem $A_1 A_{j_1} A_{j_2}$ [19] where Lemma 2 leads us to the three-qubit reduced density matrix as

$$\rho_{A_1 A_{j_1} A_{j_2}} = |\tilde{x}\rangle_{A_1 A_{j_1} A_{j_2}} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_{j_1} A_{j_2}} \langle \tilde{y}|, \quad (29)$$

with the three-qubit unnormalized states

$$\begin{aligned} |\tilde{x}\rangle_{A_1 A_{j_1} A_{j_2}} &= a|000\rangle_{A_1 A_{j_1} A_{j_2}} + b_1|100\rangle_{A_1 A_{j_1} A_{j_2}} \\ &\quad + b_{j_1}|010\rangle_{A_1 A_{j_1} A_{j_2}} + b_{j_2}|001\rangle_{A_1 A_{j_1} A_{j_2}} \\ |\tilde{y}\rangle_{A_1 A_{j_1} A_{j_2}} &= \sqrt{\sum_{k \neq j_1, j_2} |b_k|^2} |000\rangle_{A_1 A_{j_1} A_{j_2}}. \end{aligned} \quad (30)$$

The HJW theorem in Proposition 1 assures that for any pure state decomposition of $\rho_{A_1 A_{j_1} A_{j_2}}$,

$$\rho_{A_1 A_{j_1} A_{j_2}} = \sum_{h=1}^r |\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}} \langle \tilde{\phi}_h|, \quad (31)$$

where $|\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}}$ is an unnormalized state in three-qubit subsystem $A_1 A_{j_1} A_{j_2}$, there exists an $r \times r$ unitary matrix (u_{hl}) that makes a relation between pure state ensembles of $\rho_{A_1 A_{j_1} A_{j_2}}$ as

$$|\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}} = u_{h1} |\tilde{x}\rangle_{A_1 A_{j_1} A_{j_2}} + u_{h2} |\tilde{y}\rangle_{A_1 A_{j_1} A_{j_2}}. \quad (32)$$

Here we note, for each h , $|\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}}$ in Eq. (32) is a (unnormalized) superposition of a three-qubit W-class state and vacuum. Thus Lemma 1 assures that the normalized state $|\phi_h\rangle_{A_1 A_{j_1} A_{j_2}} = |\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}} / \sqrt{p_h}$ with $p_h = |\langle \tilde{\phi}_h | \tilde{\phi}_h \rangle|$ satisfies Eq. (9), that is, the three-tangle of $|\tilde{\phi}_h\rangle_{A_1 A_{j_1} A_{j_2}}$ in Eq. (22) is zero,

$$\begin{aligned} \tau \left(|\phi_h\rangle_{A_1|A_{j_1}|A_{j_2}} \right) &= \tau \left(|\phi_h\rangle_{A_1|A_{j_1}|A_{j_2}} \right) \\ &\quad - \tau \left(\rho_{A_1|A_{j_1}} \right) - \tau \left(\rho_{A_1|A_{j_2}} \right) \\ &= 0, \end{aligned} \quad (33)$$

for each h .

Eq. (33) implies that three-qubit pure state that arises in any pure state ensemble of $\rho_{A_1 A_{j_1} A_{j_2}}$ in Eq. (31) has zero as its three tangle value. Thus, from the definition of n -tangle for multi-qubit mixed state in Eq. (5), we have

$$\begin{aligned} \tau \left(\rho_{A_1|A_{j_1}|A_{j_2}} \right) &= \left[\min_{\{p_h, |\phi_h\rangle\}} \sum_h p_h \sqrt{\tau \left(|\phi_h\rangle_{A_1|A_{j_1}|A_{j_2}} \right)} \right]^2 \\ &= 0, \end{aligned} \quad (34)$$

for any the three-qubit reduced density matrix $\rho_{A_1 A_{j_1} A_{j_2}}$ of $|\psi\rangle_{A_1 A_2 \cdots A_n}$.

We now assume the induction hypothesis for Eq. (28); for any $(m-1)$ -qubit reduced density matrix $\rho_{A_1 A_{j_1} A_{j_2} \cdots A_{j_{m-2}}}$ of the generalized W-class states in Eq. (7), we assume its $(m-1)$ tangle is zero,

$$\tau \left(\rho_{A_1|A_{j_1}|A_{j_2} \cdots |A_{j_{m-2}}} \right) = 0, \quad (35)$$

and show its validity for $m \leq n-1$.

For any index vector $\vec{j} = (j_1, j_2, \dots, j_{m-1})$ with $\{j_1, j_2, \dots, j_{m-1}\} \subseteq \{2, 3, \dots, n\}$, Lemma 2 assures that

the m -qubit reduced density matrix of $|\psi\rangle_{A_1 A_2 \dots A_n}$ on subsystems $A_1 A_{j_1} \dots A_{j_{m-1}}$ is

$$\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} = |\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{x}| + |\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{y}|, \quad (36)$$

where $|\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ and $|\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ are the m -qubit unnormalized states in Eq. (20). By HJW theorem in Proposition 1, we note that any pure state decomposition

$$\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} = \sum_{h=1}^r |\tilde{\phi}_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{\phi}_h|, \quad (37)$$

is related with Eq. (36) by some $r \times r$ unitary matrix

(u_{hl}) such that

$$|\tilde{\phi}_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} = u_{h1} |\tilde{x}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} + u_{h2} |\tilde{y}\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}, \quad (38)$$

for each h . Furthermore, the normalization $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} = |\tilde{\phi}_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} / \sqrt{p_h}$ with $p_h = |\langle \tilde{\phi}_h | \tilde{\phi}_h \rangle|$ is a superposition of a m -qubit generalized W-class state and vacuum, which is again a generalized W-class state.

From the definition of pure state tangle in Eq. (22), the m tangle of each m -qubit pure state $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ is

$$\tau(|\phi_h\rangle_{A_1 | A_{j_1} | \dots | A_{j_{m-1}}}) = \tau(|\phi_h\rangle_{A_1 | A_{j_1} \dots A_{j_{m-1}}}) - \sum_{k=2}^{m-1} \sum_{\vec{i}^k} \tau(\rho_{A_1 | A_{i_1} | \dots | A_{i_{k-1}}}^h)^{k/2}, \quad (39)$$

where $\rho_{A_1 A_{i_1} \dots A_{i_{k-1}}}^h$ is the reduced density matrix of $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ on k -qubit subsystems $A_1 A_{i_1} \dots A_{i_{k-1}}$ with the index vector $\vec{i}^k = (i_1, i_2, \dots, i_{k-1})$ for

$\{i_1, i_2, \dots, i_{k-1}\} \subseteq \{j_1, j_2, \dots, j_{m-1}\}$. Let us further divide the last term of the inequality into the summation of two tangles and the others;

$$\tau(|\phi_h\rangle_{A_1 | A_{j_1} | \dots | A_{j_{m-1}}}) = \tau(|\phi_h\rangle_{A_1 | A_{j_1} \dots A_{j_{m-1}}}) - \sum_{l=1}^{m-1} \tau(\rho_{A_1 | A_{j_l}}^h) - \sum_{k=3}^{m-1} \sum_{\vec{i}^k} \tau(\rho_{A_1 | A_{i_1} | \dots | A_{i_{k-1}}}^h)^{k/2}. \quad (40)$$

For each $k = 3, \dots, m-1$, $\rho_{A_1 A_{i_1} \dots A_{i_{k-1}}}^h$ in the last summation of Eq. (40) is a k -qubit reduced density matrix of the generalized W-class state $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$, therefore the induction hypothesis assures that its k tangle is zero;

$$\tau(\rho_{A_1 | A_{i_1} | \dots | A_{i_{k-1}}}^h) = 0, \quad (41)$$

for each $k = 3, \dots, m-1$ and index vector $\vec{i}^k = (i_1, i_2, \dots, i_{k-1})$. Furthermore, Lemma 1 implies that the usual monogamy inequality in terms of one and two tangles is saturated by $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$;

$$\tau(|\phi_h\rangle_{A_1 | A_{j_1} \dots A_{j_{m-1}}}) = \sum_{l=1}^{m-1} \tau(\rho_{A_1 | A_{j_l}}^h), \quad (42)$$

for each h .

Eq. (41) together with Eq. (42) imply that

$$\tau(|\phi_h\rangle_{A_1 | A_{j_1} | \dots | A_{j_{m-1}}}) = 0 \quad (43)$$

for each $|\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}}$ that arises in the decomposition of $\rho_{A_1 A_{j_1} \dots A_{j_{m-1}}}$,

$$\begin{aligned} \rho_{A_1 A_{j_1} \dots A_{j_{m-1}}} &= \sum_{h=1}^r |\tilde{\phi}_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \tilde{\phi}_h| \\ &= \sum_{h=1}^r p_h |\phi_h\rangle_{A_1 A_{j_1} \dots A_{j_{m-1}}} \langle \phi_h|. \end{aligned} \quad (44)$$

Thus, from the definition of n -tangle for multi-qubit mixed state in Eq. (5), we have

$$\tau\left(\rho_{A_1|A_{j_1}|\dots|A_{j_{m-1}}}\right)=\left[\min_{\{p_h,|\phi_h\rangle\}}\sum_h p_h\sqrt{\tau\left(|\phi_h\rangle_{A_1|A_{j_1}|\dots|A_{j_{m-1}}}\right)}\right]^2=0, \quad (45)$$

for any the m -qubit reduced density matrix $\rho_{A_1A_{j_1}\dots A_{j_{m-1}}}$ of $|\psi\rangle_{A_1A_2\dots A_n}$ with $3 \leq m \leq n-1$. Now Eq. (45) together with Lemma 1, we have the following theorem showing the saturation of multi-qubit SM inequality by generalized W-class states.

Theorem 3. *The strong monogamy inequality of multi-qubit entanglement is saturated by the generalized W-class states;*

$$\tau\left(|\psi\rangle_{A_1|A_2\dots A_n}\right)=\sum_{m=2}^{n-1}\sum_{\vec{j}_m}\tau\left(\rho_{A_1|A_{j_1^m}|\dots|A_{j_{m-1}^m}}\right)^{m/2}, \quad (46)$$

for any multi-qubit generalized W-class state in Eq. (7).

IV. CONCLUSIONS

We have considered a large class of multi-qubit generalized W-class states, and provided a strong evidence for SM inequality of multi-qubit entanglement. Although

providing an analytical proof of SM inequality for arbitrary multi-qubit states seems to be a formidable challenge because there are numerous optimization processes arising in the recurrent definition of n -tangle, we have successfully resolved this problem by investing the structural properties of W-class states, and analytically shown that strong monogamy inequality is saturated by this class of states.

Our result characterizes the strongly monogamous nature of arbitrary multi-qubit W-class states. Noting the importance of the study on multipartite entanglement, our result can provide a rich reference for future work on the study of entanglement in complex quantum systems.

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